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ON THE ASYMPTOTIC SOLUTIONS OF THE EQUATIONS OF MOTION OF MECHANICAL SYSTEMS*

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The existence of solutions of the equations of motion of non-natural mechanical systems of a certain form which tend to a position of equilibrium when the time increases without limit is proved by the methods described in /1, 2/. The corresponding instability theorem is proved.

1. Let us consider the motion of a mechanical system which is described by the equations

$$\frac{d}{dt}\left(\frac{\partial R}{\partial q^{\cdot}}\right)-\frac{\partial R}{\partial q}=0,\quad q\in\mathbf{R}^n \tag{1.1}$$
$$R:\mathbf{R}^n\{q^{\cdot}\}\times\mathbf{R}^n\{q\}\to\mathbf{R}$$
$$R=R_2+R_1+R_0,\quad R_2={}^1/_2\langle K(q)\,q^{\cdot},q^{\cdot}\rangle$$

where R_i are homogeneous forms of generalized velocities of degree i. We shall treat (1.1) as the equations of motion of a reduced system which is obtained from a certain initial system by eliminating the cyclic coordinates, and R is a Routh function which depends on the cyclic constants, K is a positive-definite symmetric matrix and $\langle,\rangle$ is a scalar product in $\mathbf{R}^n$. It may be assumed without any loss in generality that $R_1=\langle l(q),q^{\cdot}\rangle$, l: $\mathbf{R}^n\to\mathbf{R}^n$ is a vector field and $l(0)=0$ /3/. Let the reduced force function $R_0=V$, the components of the matrix K and of the vector field l be analytic functions of the generalized coordinates (we shall assume the values of the cyclic integrals to be constant).

Let us assume that the origin of coordinates $q=0$ is the equilibrium position of system (1.1) $V'(0)=0$ and $V(0)=0$. We shall say that a solution of (1.1) $q(t)\not\equiv 0$ is asymptotic if $q(t)\to 0$ as $t\to+\infty$. If the function V is absolutely negative in a certain neighbourhood of the equilibrium $q=0$, the position of equilibrium is stable (Routh's theorem /4/) and there are no asymptotic solutions.

2. Suppose

$$V(q)=\sum_{k\geqslant m+1}^{\infty}v_k(q),\quad m\geqslant 1$$
$$l(q)=\sum_{k\geqslant s+1}^{\infty}l_k(q),\quad s\geqslant 0$$

where v_k and the components l_k are homogeneous forms of degree k.

Theorem. If v_{m+1} can take positive values and $s>[(m-1)/2]$, a solution of Eq.(1.1) exists which is asymptotic to the position of equilibrium $q=0$ and this position of equilibrium is unstable.

In the case when $m=1$, the proof of instability constitutes Salvadori's theorem /5/ and the existence of an asymptotic solution follows from the Lyapunov theorem on conditional asymptotic stability /6/.

Let us prove the theorem for the case when $m>1$. We shall write the equation of motion in "geodesic" form

$$q^{\cdot\cdot}=\Gamma(q,q^{\cdot})+\Omega(q)\,q^{\cdot}+f(q) \tag{2.1}$$
$$\Gamma:\ \mathbf{R}^n\ \{q\}\times\mathbf{R}^n\ \{q^{\cdot}\}\to\mathbf{R}^n$$
$$\Omega:\ \mathbf{R}^n\ \{q\}\to\mathbf{R}^{n2}$$
$$\Omega(q)=K^{-1}(q)\left(\left(\frac{\partial}{\partial q}(l(q))\right)^T-\frac{\partial}{\partial q}(l(q))\right)$$

where $\partial l/\partial q$ is the Jacobian matrix of the field l, and Γ is a mapping which is quadratic with respect to $q^{\cdot}$, the coefficients of which are the Christoffel symbols of the metric $d\sigma^2=$ ${}^1/_2\langle K(q)\,dq,dq\rangle$. The expansion of $\Omega(q)$ in a Maclaurin series has the form

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$$\Omega(q)=\sum_{k\geqslant s}^{\infty}\Omega_k(q)$$

Let the system be written in normal coordinates ($K(0)=E$, E is a unit matrix). The expansion of $f(q)$ then has the form

$$f(q)=v_{m+1}'(q)+\sum_{k\geqslant m+1}^{\infty}f_k(q)$$

3. After substitution of the independent variable $\tau=\ln t$, Eqs.(2.1) are written in the form (here the dots signify differentiation with respect to τ)

$$F(q,\tau)\equiv q^{\cdot\cdot}-(q^{\cdot}+\Gamma(q,q^{\cdot})+e^{\tau}\Omega(q)\,q^{\cdot}+e^{2\tau}f(q))=0 \tag{3.1}$$

We shall seek a solution of Eqs.(3.1) in the form of the series

$$q(\tau)=\sum_{(m-1)j\leqslant k}^{\infty}q_{kj}e^{-\zeta_k\tau}\tau^j \tag{3.2}$$
$$\zeta_k=(2+k)/(m-1),\quad q_{kj}\in\mathbf{R}^n,\quad k,j\in\mathbf{N}\cup\{0\}$$

and we shall seek the coefficients q_{kj} with the help of induction which increases with respect to k and decreases with respect to j. Let us write

$$q(\tau)=\sum_{k}^{\infty}a_ke^{-\zeta_k\tau},\quad q^{\cdot}(\tau)=\sum_{k}^{\infty}b_ke^{-\zeta_k\tau}$$
$$q^{\cdot\cdot}(\tau)=\sum_{k}^{\infty}c_ke^{-\zeta_k\tau}$$
$$a_k=\sum_{(m-1)j\leqslant k}q_{kj}\tau^j,\quad b_k=\sum_{(m-1)j\leqslant k}q_{kj}\tau^{j-1}$$
$$(-\zeta_k\tau+j);\quad c_k=\sum_{(m-1)j\leqslant k}q_{kj}\tau^{j-2}(\zeta_k^2\tau^2-2j\zeta_k\tau+j(j-1))$$

We substitute these expressions into (3.1) and equate the coefficients of $e^{-\zeta_k\tau}$. We specify that

$$q_k=\sum_{i\leqslant k-2}a_ie^{-\zeta_i\tau},\quad x_k=F(q_k,\tau)$$

and substitute $q_2=q_{00}e^{-\zeta_0\tau}$ into (3.1) and collect the terms in $e^{-\zeta_0\tau}$. The function $\Gamma(q_2,q_2^{\cdot})$ does not yield terms with $e^{-\tau}$ to a power which is less than ζ_2. By virtue of the fact that the inequality $s>[(m-1)/2]$ is satisfied, the minimal power of $e^{-\tau}$ which the function $e^{\tau}\Omega(q_2)\,q_2^{\cdot}$ yields is equal to ζ_1. In order to find q_{00} we have the equation

$$\nu_0q_{00}=v_{m+1}'(q_{00}),\quad \nu_0=\frac{2(m+1)}{(m-1)^2}$$

Let e_0 be the point of the maximum of v_{m+1} on a unit sphere. According to the condition this maximum is positive and, consequently,

$$v_{m+1}'(e_0)=\varkappa e_0,\ \varkappa>0$$

and the above-mentioned equation can be satisfied by putting $q_{00}=de_0$, where $d=(\nu_0/\varkappa)^{1/(m-1)}$. When this is done, x_2 contains terms with $e^{-\tau}$ to powers not less than ζ_1.

Let us now estimate the eigenvalues of the operator $v_{m+1}''(q_{00})$. According to Euler's theorem on homogeneous functions

$$v_{m+1}''(q_{00})\,e_0=d^{m-1}v_{m+1}''(e_0)\,e_0=(\nu_0m/\varkappa)\,v_{m+1}'(e_0)=\nu_0me_0$$

Since e_0 is the point of the maximum of v_{m+1} on the unit sphere, the characteristic numbers of the operator $v_{m+1}''(q_{00})$ which acts on the invariant subspace $e_0^{\perp}$ will be non-positive ($e_0^{\perp}$ is a subsapace perpendicular to e_0).

Let the coefficients $a_0,\ldots,a_{k-2}$, $b_0,\ldots,b_{k-2}$, $c_0,\ldots,c_{l-2}$ be found and let $x_k=F(q_k,\tau)$ contain terms with $e^{-\tau}$ to powers not less than ζ_{k-1}. We find a_{k-1}, b_{k-1} and c_{k-1} by substituting q_{k+1} into (3.1) and collecting the terms in $e^{-\zeta_{k-1}\tau}$. The expressions $\Gamma(q_{k+1},q_{k+1}^{\cdot})$ and $e^{\tau}\Omega(q_{k+1})$ q_{k+1} $(k-1)$-x do not yield coefficients. These coefficients satisfy the equation

$$c_{k-1}-b_{k-1}-v_{m+1}''(q_{00})\,a_{k-1}=\Psi_{k-1} \tag{3.3}$$

where Ψ_{k-1} is a certain "polynomial" of $a_0,\ldots,a_{k-2}$, $b_0,\ldots,b_{k-2}$, $c_0,\ldots,c_{k-2}$.

Let $1<k<m$, then $a_{k-1}=q_{(k-1)0}$, $b_{k-1}=-\zeta_{k-1}q_{(k-1)0}$, $c_{k-1}=\zeta_{k-1}^2q_{(k-1)0}$. Eq.(3.3) takes the form

$$\nu_{k-1}q_{(k-1)0}-v_{m+1}''(q_{00})\,q_{(k-1)0}=\Psi_{k-1} \tag{3.4}$$
$$\nu_k=\frac{(k+2)(m+k+1)}{(m-1)^2},\quad \nu_0m=\nu_{m-1}$$

It is obvious that the quantity Ψ_{k-1} is independent of τ. When $1 < k < m$, the matrix $(\nu_{k-1} E - v''_{m+1}(q_{00}))$ is non-singular and $q_{10}, \ldots, q_{(m-2)0}$ are therefore found uniquely.

Let $k > m$. It can be shown that the maximum power of τ which is contained in Ψ_{k-1} is equal to $[(k-1)/(m-1)]$. Let us represent Ψ_{k-1} in the form

$$\Psi_{k-1} = \sum_{(m-1)j \leqslant k-1} \Psi_{k-1}^{(j)} \tau^j$$

and, by equating terms in (3.3) with the same powers of τ we obtain

$$\begin{gathered}\nu_{k-1} q_{(k-1)j} - v''_{m+1}(q_{00}) q_{(k-1)j} - \eta_{k-1} q_{(k-1)(j+1)} + \\ (j+1)(j+2) q_{(k-1)(j+2)} = \Psi_{k-1}^{(j)} \\ \eta_k = \frac{(2k+m+3)(j+1)}{m-1}, \quad j \leqslant l, \quad l = [(k-1)/(m-1)] \\ q_{(k-1)(l+1)} = q_{(k-1)(l+2)} = 0\end{gathered} \tag{3.5}$$

$\Psi_{(k-1)}^{(j)}$ are solely dependent on the coefficients q_{rs} $(m-1)s \leqslant r < k-1$. Since the matrix $(\nu_{k-1} E - v''_{m+1}(q_{00}))$ is non-singular when $k > m$, by using induction which decreases with respect to j, it is possible to find the coefficients $q_{(k-1)j}$ uniquely.

Finally, let us consider the case when $k = m$. The quantity Ψ_{m-1} does not contain power of τ and one may therefore write

$$\Psi_{m-1} = \Psi_{m-1}^{(0)} = \alpha e_0 + f, \quad \alpha \in \mathbf{R}, \quad f \in e_0^{\perp}$$

To determine $q_{(m-1)0}, q_{(m-1)1}$, we then have the system of equations

$$\begin{gathered}\nu_{m-1} q_{(m-1)0} - v''_{m+1}(q_{00}) q_{(m-1)0} - \eta_{m-1} q_{(m-1)1} = \alpha e_0 + f \\ \nu_{m-1} q_{(m-1)1} - v''_{m+1}(q_{00}) q_{(m-1)1} = 0\end{gathered} \tag{3.6}$$

By putting $q_{(m-1)1} = -(\alpha/\eta_{m-1}) e_0$, we satisfy the second equality and we solve the first equality in the subspace $e_0^{\perp}$ where the operator $(\nu_{m-1} E - v''_{m+1}(q_{00}))$ is non-singular.

4. Let us now consider the Banach spaces $\mathbf{E}_{\tau_0, \alpha}^{(0)}, \mathbf{E}_{\tau_0, \alpha}^{(2)}$: $\mathbf{E}_{\tau_0, \alpha}^{(2)}$ is the space of the functions q: $[\tau_0, +\infty) \to \mathbf{R}$ which are continuously doubly differentiable in the infinite semi-interval $[\tau_0, +\infty)$, $\tau_0 > 0$ and for which the norm

$$\| q \|_{\tau_0, \alpha}^{(2)} = \sup_{\tau \geqslant \tau_0} \{ e^{\alpha\tau} [\, | q''(\tau) | + | q'(\tau) | + | q(\tau) | \,] \}, \quad \alpha > 0$$

is finite. $\mathbf{E}_{\tau_0, \alpha}^{(0)}$ is the space of the functions p: $[\tau_0, +\infty) \to \mathbf{R}$ which are continuous in $[\tau_0, +\infty)$ and for which the norm

$$\| p \|_{\tau_0, \alpha}^{(0)} = \sup_{\tau \geqslant \tau_0} \{ e^{\alpha\tau} | p(\tau) | \}$$

is finite and the differential operator

$$D : \mathbf{E}_{\tau_0, \alpha}^{(2)} \to \mathbf{E}_{\tau_0, \alpha}^{(0)}; \quad D = \frac{d^2}{d\tau^2} + a \frac{d}{d\tau} + b, \quad a, b \in \mathbf{R}$$

Lemma. Let $\operatorname{Re} \mu_1, \mu_2 > -\alpha$, where μ_1 and μ_2 are the roots of the equation

$$\mu^2 + a\mu + b = 0$$

Then, the operator D has a bounded inverse and, for any $p \in \mathbf{E}_{\tau_0, \alpha}^{(0)}$

$$\| D^{-1} p \|_{\tau_0, \alpha}^{(2)} \leqslant C \| p \|_{\tau_0, \alpha}^{(0)} \tag{4.1}$$

and the constant $C > 0$ is solely dependent on a, b, α and independent of τ_0.

Proof. By using the method of the variation of constants to solve the differential equation with the "initial" conditions

$$\begin{gathered} q'' + aq' + b = p(\tau), \quad p \in \mathbf{E}_{\tau_0, \alpha}^{(0)} \\ q(+\infty) = q'(+\infty) = 0 \end{gathered} \tag{4.2}$$

it is possible to obtain the explicit formulae

$$\begin{gathered} q(\tau) = (\mu_2 - \mu_1)^{-1} \{ e^{\mu_1 \tau} I(\tau, \mu_1) - e^{\mu_2 \tau} I(\tau, \mu_2) \} \\ \mu_1 \neq \mu_2, \quad \mu_1, \mu_2 \in \mathbf{R} \\ q(\tau) = e^{\mu\tau} \int_{\tau}^{+\infty} I(s, \mu)\, ds, \quad \mu_1 = \mu_2 = \mu, \quad \mu \in \mathbf{R} \\ q(\tau) = \omega^{-1} e^{\gamma\tau} \{ \cos \omega\tau I_1(\tau) - \sin \omega\tau I_2(\tau) \} \\ \mu_{1,2} = \gamma \pm \sqrt{-1}\,\omega, \quad \gamma, \omega \in \mathbf{R} \end{gathered} \tag{4.3}$$

$$I(\tau, \mu) = \int_{\tau}^{+\infty} e^{-\mu s} p(s)\, ds$$

$$I_1(\tau) = \int_{\tau}^{+\infty} e^{-\gamma s} \sin \omega s p(s)\, ds$$

$$I_2(\tau) = \int_{\tau}^{+\infty} e^{-\gamma s} \cos \omega s p(s)\, ds$$

By virtue of the fact that $\operatorname{Re} \mu_1, \mu_2 > -\alpha$, all the integrals in (4.3) converge. By differentiating relationships (4.3) and majorizing the corresponding integrals by exponentially decaying functions, we obtain the estimates

$$\| q \|^{(2)}_{\tau_0, \alpha} \leqslant \left\{1 + |\mu_2 - \mu_1|^{-1} \left[\frac{\mu_1^2 + \mu_1 + 1}{\alpha + \mu_1} + \frac{\mu_2^2 + \mu_2 + 1}{\alpha + \mu_2} \right]\right\} \| p \|^{(0)}_{\tau_0, \alpha}, \quad \mu_1 \neq \mu_2, \quad \mu_1, \mu_2 \in \mathbf{R} \tag{4.4}$$

$$\| q \|^{(2)}_{\tau_0, \alpha} \leqslant \left\{1 + \frac{2|\mu| + 1}{\alpha + \mu} + \frac{\mu^2 + \mu + 1}{(\alpha + \mu)^2}\right\} \| p \|^{(0)}_{\tau_0, \alpha}\, \mu_1 = \mu_2 = \mu, \quad \mu \in \mathbf{R}$$

$$\| q \|^{(2)}_{\tau_0, \alpha} \leqslant \{1 + 2(|\omega|(\alpha + \gamma))^{-1} [|\gamma^2 - \omega^2| + 2|\omega\gamma| + |\gamma| + |\omega| + 1]\} \| p \|^{(0)}_{\tau_0, \alpha}$$

$$\mu_{1,2} = \gamma \pm \sqrt{-1}\omega, \ \gamma, \ \omega \in \mathbf{R}$$

for the solution of (4.2) since the modulus of the integral $I(\tau, \mu)$ is majorized, for example, by the function $\| p \|^{(0)}_{\tau_0, \alpha} (\alpha + \mu)^{-1} e^{-(\alpha+\mu)\tau}$.

The operator D therefore has a bounded inverse and (4.1) follows from (4.4).

If series (3.2) converges, an asymptotic solution of (3.1) exists. We shall find the solution of Eqs. (3.1) in the form $q = q_m + y$, where q_m is the approximate solution found on the $(m-1)$-th step and y is an unknown series of the form of (3.2) which is uniformly convergent in a certain infinite semi-interval $[\tau_0, +\infty)$, $\tau_0 > 0$ when $k > m - 1$.

Let us now consider three examples of the spaces of such series. $\mathbf{B}^{(l)}_{\tau_0, \alpha}$ are the C^l-spaces of the vector functions $y: [\tau_0, +\infty) \to \mathbf{R}^n$ which are representable in the form of the series (3.2) when $k > m - 1$ with the norms

$$\| y \|^{(l)}_{\tau_0, \alpha} = \sup_{\tau \geqslant \tau_0} \{e^{\alpha\tau} [\sum_{r=0}^{l} \| y^{(r)}(\tau) \|]\}$$

$$l = 0, 1, 2, \quad \alpha = (m + 1 + \delta)/(m - 1), \quad 0 < \delta \leqslant 1$$

$$\mathbf{B}^{(0)}_{\tau_0, \alpha} \supset \mathbf{B}^{(1)}_{\tau_0, \alpha} \supset \mathbf{B}^{(2)}_{\tau_0, \alpha}$$

We now consider the differential operator

$$D: \mathbf{B}^{(2)}_{\tau_0, \alpha} \to \mathbf{B}^{(0)}_{\tau_0, \alpha}; \quad D = \frac{d^2}{d\tau^2} - \frac{d}{d\tau} - \ddot{v}_{m+1}(q_{00})$$

and expand $\mathbf{R}^n$ as an orthogonal sum of the characteristic uni-dimensional subspaces of the operator $\ddot{v}_{m+1}(q_{00})$. Let P_λ be the projector onto the subspace which corresponds to an eigenvalue λ:

$$P_\lambda D = D_\lambda P_\lambda; \quad D_\lambda = \frac{d^2}{d\tau^2} - \frac{d}{d\tau} - \lambda$$

Let us now consider the roots of the equation $\mu^2 - \mu - \lambda = 0$. When $\lambda \leqslant -1/4$, we have $\operatorname{Re} \mu_1, \mu_2 = 1/2$ and, when $-1/4 < \lambda \leqslant 0$, the roots are real and belong to the interval $[0; 1]$. In the case of the operator $\ddot{v}_{m+1}(q_{00})$, there is only a single positive characteristic number $\lambda = v_{m-1} = mv_0$. In this case $\mu_1 = 2m/(m-1)$, $\mu_2 = -(m+1)/(m-1)$. In any case $\operatorname{Re} \mu_1, \mu_2 > -\alpha$, $\alpha = (m + 1 + \delta)/(m - 1)$ and, by virtue of the lemma, the operator D_λ is invertible and

$$\| D_\lambda^{-1} p \|^{(1)}_{\tau_0, \alpha} \leqslant \| D_\lambda^{-1} p \|^{(2)}_{\tau_0, \alpha} \leqslant C(\lambda) \| p \|^{(0)}_{\tau_0, \alpha}$$

Since the space $\mathbf{R}^n$ is finite-dimensional, it is possible to make the given estimate uniform with respect to λ. Consequently, the operator D is invertible and, for any $p \in \mathbf{B}^{(0)}_{\tau_0, \alpha}$

$$\| D^{-1} p \|^{(1)}_{\tau_0, \alpha} \leqslant C \| p \|^{(0)}_{\tau_0, \alpha}$$

where $C > 0$ is independent of τ_0.

Let us write (3.1) in the form of a functional equation

$$y = D^{-1}\Phi(y) \tag{4.5}$$

$$\Phi(y) = -v''_{m+1}(q_{00})\,y + (\Gamma(q, q^{\cdot}) - \Gamma(q_m, q_m^{\cdot}) +$$
$$e^{\tau}(\Omega(q)\,q^{\cdot} - \Omega(q_m)\,q_m^{\cdot}) + e^{2\tau}(f(q) - f(q_m))) - x_m$$

By using the technique in /2/, it is possible to obtain the estimate

$$\| \Phi(y_1) - \Phi(y_2) \|_{\tau_0,\alpha}^{(0)} \leqslant C_1(\tau_0) \| y_1 - y_2 \|_{\tau_0,\alpha}^{(1)} \tag{4.6}$$

Also, $C_1(\tau_0) \to 0+$ as $\tau_0 \to +\infty$ and y_1 and y_2 belong to a sphere of radius L in $\mathbf{B}_{\tau_0,\alpha}^{(1)}$, where $L = 2\| x_m \|_{\tau_0,\alpha}^{(1)}$. Consequently, for sufficiently large $\tau_0\, D^{-1}\Phi(y)$, it will be a compacting operator on a certain sphere in $\mathbf{B}_{\tau_0,\alpha}^{(1)}$. Hence $D^{-1}\Phi(y)$ has a fixed point /7/ i.e. Eq.(4.5) has a solution in the form of the convergent series (3.2) when $k > m - 1$ which also proves the convergence of the formal solution which has been constructed earlier.

Eqs.(2.1) then have a particular asymptotic solution in the form of the uniformly converging series

$$q(t) = \sum_{(m-1)j \leqslant k}^{\infty} q_{kj} t^{-t_k} \ln^j t \tag{4.7}$$

5. The time substitution $t \to c - t$ transforms the system being considered with a Routh function $R = R_2 + R_1 + R_0$ into a system with a Routh function $R^- = R_2 - R_1 + R_0$. However, the conditions of the theorem are invariant with respect to such a substitution and the asymptotic solution will therefore also exist in the case of a system with R^- whence it follows that trajectories will exist for the initial system which emerge onto the boundary of a sphere of fixed radius from a neighbourhood of the equilibrium which is as small as may be desired over a finite time, that is, there is instability. The theorem is thereby completely proved.

This theorem is an extension of the theorems /1, 2/ concerning the existence of asymptotic trajectories on unnatural systems. We note that, under the conditions of the theorem, constancy of the sign of the part of the Hamiltonian function which is independent of momenta has not been stipulated and this theorem therefore generalizes the result in /8/ in a known sense.

A stability theorem has been formulated in /9/, the proof of which, presented in /9/, contains a number of inaccuracies although, when judged as a whole, the theorem is true.

Let $H(p, q)$ be the Hamiltonian function of a certain mechanical system which satisfies the conditions of the theorem in /9/. It may then be shown that $H_0(q) = H(0, q) < 0$ for any $q \in V_\varepsilon^+$, where $V_\varepsilon^+ = \{q \in \mathbf{R}^n\colon \| q \| < \varepsilon,\ V(q) > 0\}$ is non-void and connected.

We shall now show that systems exist which do not satisfy the conditions of the theorem in /9/ but which satisfy the conditions of the theorem in this paper.

Let us consider a system with a Routh function

$$R(q_1, q_2, q_1^{\cdot}, q_2^{\cdot}) = {}^1/_2\,(q_1^{\cdot 2} + q_2^{\cdot 2}) + q_1^2(q_1 q_2^{\cdot} - q_2 q_1^{\cdot}) + {}^1/_2 q_2^3$$

Here $q_1 = q_2 = 0$ is the position of equilibrium, $V(q_1, q_2) = v_3(q_1, q_2) = {}^1/_2 q_2^3$ takes positive values when $q_2 > 0$ and the order of smallness of the terms accompanying $q_1^{\cdot}, q_2^{\cdot}$ in R_1 is equal to three, that is, there is instability according to the theorem presented in this paper.

The Hamiltonian function of the system under consideration is equal to

$$H(p_1, p_2, q_1, q_2) = {}^1/_2\,(p_1^2 + p_2^2) - q_1^2(p_2 q_1 -$$
$$p_1 q_2) + H_0(q_1, q_2),\ \ 2H_0(q_1, q_2) = q_1^4(q_2^2 + q_1^2) - q_2^3$$

It is obvious that $H_0(q_1, q_2)$ can take positive values when $q_2 > 0$ in a neighbourhood of $q_1 = q_2 = 0$ which may be as small as may be desired, that is, this system does not satisfy the conditions of the theorem in /9/.

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ON A MEASURE OF THE CLOSENESS OF NEUTRAL SYSTEMS TO INTERNAL RESONANCE*

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For certain classes of parametrically perturbed resonance systems that are neutral in a linear approximation, a quantitative characteristic is introduced for the closeness of the system of resonance: the magnitude of the critical detuning value for resonance δ^* at which the change in stability occurs as the system withdraws from resonance. The problem of finding this critical value is made complicated by the non-linear nature of the change in stability in neutral systems. It is solved below for third-order resonances in a situation that guarantees the passage of instability into asymptotic stability as the system withdraws from resonance.

Knowledge of the quantity δ^* enables the strong instability domain /1, 2/ in parameter space to be estimated, enables the danger of resonance to be characterized, and enables the structural parameter in the system, the shift of the resonance phases, to be clarified, whose variation would enable the danger of resonance to be increased or reduced.

1. Formulation of the problem. Fundamental assumptions. In the l-dimensional real space R^l we consider the system of differential equations that depends continuously on the parameter $\mu \in D$

$$z^{\cdot} = A(\mu) z + \sum_{j=k-1\geqslant 2}^{\infty} F^{(j)}(\mu, t, z) \tag{1.1}$$

where $D \subseteq R^d$ is a certain closed d-dimensional domain containing the origin, and $F^{(j)}$ are l-dimensional vector forms of j-th order whose coefficients are almost periodic functions of t uniformly in $\mu \in D$.

Let the matrix $A(\mu)$ have n pairs of different purely imaginary eigenvalues $\pm i\nu_s(\mu), s = 1, \ldots, n$ in D while the remaining eigenvalues have negative real parts in D.

Retaining the definitions from /3/, we consider (1.1) to be an F-system and there is a k-th -order resonance therein for $\mu = 0$ ($m_s \geqslant 0$ are integers):

$$\lambda = \langle m, \ \nu(0)\rangle \in N_2'^k, \quad m = (m_1, \ldots, m_n), \quad k = |m| = m_1 + \ldots + m_n. \tag{1.2}$$

The concepts of the F-system and the set $N_2'^k$ are described in detail in /3/. We recall that the continuous normal form of F-systems is reducible to autonomous form while $N_2'^k$ is contained in the minimum modulus generated by the spectrum of the non-linearity coefficients.

We will confine ourselves to studying a purely critical system when $l = 2n$. The case $l > 2n$ reduces to it by using the reduction principle /4/.

In addition to the initial parameters it is convenient to introduce the parameters $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ and the resonance detuning δ by setting

$$\varepsilon_s(\mu) = \nu_s(\mu) - \nu_s(0), \quad \delta(\mu) = \langle m, \varepsilon(\mu)\rangle$$

The equation $\delta = 0$ defines a certain k-resonance surface Γ_k in D.

The k-resonance surface is mapped into a k-resonance plane Π_k: $\langle m, \varepsilon(\mu)\rangle = 0$ in the

*Prikl.Matem.Mekhan., 50, 6, 945-953, 1986